PRECALCULUS EXPLORATION

BASIC LIMIT & DERIVATIVE RULES

In part 3 you learned what a derivative is and how it is calculated. Calculating derivatives makes up a majority of what you will be doing in Calculus however; calculating them using the definition of a derivative can be rather time-consuming. So, in this lesson we will be searching for short-cuts to aide us in calculating these derivatives.

We will still be using the definition of a derivative in many parts of this lesson. So, when we need to do so we will be using the 1st definition of a derivative as given in part 3.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

As we use this definition, it is important to note that ***h* is the variable in the limit**. That means any part of the expression which contains *h* will be treated as if it were a variable and, thus, will be simplified.

On the other hand, any part of the limit that does not contain *h*, even if it contains another variable like *x*, will be treated as a **constant** for algebraic calculations.

Keep these things in mind as we proceed.

SECTION A: DERIVATIVE OF A CONSTANT

As we work through this section, we will be referring to the following functions:

$$f(x) = 8 \qquad g(x) = -\frac{3}{4} \qquad j(x) = \pi$$

Notice that these functions are constant functions in that they have no variable on the right of the equal sign. Based on this, determine the value of the following.

1. $f(2) =$ **2.** $g(5) =$ **3.** $j(-1) =$

4. $f(-6) =$ **5.** $g(-1.6) =$ **6.** $j\left(-\frac{3}{2}\right) =$

7. $f(c) =$ **8.** $g(c) =$ **9.** $j(c) =$

10. $f(x+h) =$ **11.** $g(x+h) =$ **12.** $j(x+h) =$

Based on the results of the previous page, do constant functions ever change in value as *x* (or whatever letter is used for the domain) changes in value?

Now find the following (using the definition of a derivative).

13. $f'(x) =$

14. $g'(x) =$

15. $j'(x) =$

What do you notice about the values of all of these derivatives?

Based on your findings, fill in the derivative rule below.

DERIVATIVE OF A CONSTANT

If $f(x) = c$, where c is a constant, then $f'(x) =$ __________.

<u>SECTION B</u>: DERIVATIVE OF POWER

In order to discover this rule, you will need to find the derivative of the following functions using the definition of a derivative.

1. $f(x) = x$

2. $g(x) = x^2$

3. $j(x) = x^3$

4. $k(x) = x^4$

If you calculated the derivatives on the previous page correctly, you should see a pattern develop [particularly in #2 − 4]. Based on the pattern you see, write the derivatives of the following.

5. $f\left(x\right)=x^5$

6. $f\left(x\right)=x^6$

7. $f\left(x\right)=x^7$

8. $f\left(x\right)=x^8$

9. $f\left(x\right)=x^9$

10. $f\left(x\right)=x^{10}$

Based on your findings, complete the derivative rule below.

THE POWER RULE

Given $f\left(x\right)=x^n$, where n is a real-number exponent, then $f'\left(x\right)=$__________.

We have seen this rule being used when there are exponents that are positive integers, however, the rule as stated above, says it should work for any real-numbered exponent. Let’s look some other situations.

11. $f(x) = \frac{1}{x}$

(A) Use the definition of a derivative to find $f'(x)$

(B) Based on a rule of exponents, we know that $x^{-n} = \frac{1}{x^n}$. Therefore, $f(x) = \frac{1}{x} = x^{-1}$.

Use this fact and the power rule from the previous page now to find $f'(x)$ and verify that you obtain the same result as in (A).

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12. $f(x) = \sqrt{x}$

(A) Use the definition of a derivative to find $f'(x)$

(B) Based on a rule of exponents, we know that $\sqrt[n]{x^m} = x^{m/n}$. Therefore, $f(x) = \sqrt{x} = x^{1/2}$.

Use this fact and the power rule from the previous page now to find $f'(x)$ and verify that you obtain the same result as in (A).

PRACTICE

Find the derivatives of the following using the power rule.

1. $f(x) = \dfrac{1}{x^3}$

2. $f(x) = \sqrt[3]{x}$

3. $f(x) = \sqrt[4]{x^3}$

4. $f(x) = \dfrac{1}{\sqrt[4]{x}}$

<u>SECTION C</u>: NEW DERIVATIVE RULES FROM OLD

Before we look at the derivative rules that will be discussed in this section, we need to look at some limit rules first.

Given that a and c are both constants and $f(x)$ and $g(x)$ are both functions.

Sum Rule for Limits	$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$
Difference for Limits:	$\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$
Constant Multiple Rule For Limits:	$\lim_{x\to a} c\cdot f(x)=c\cdot\lim_{x\to a} f(x)$

The rules above are easily verified and do not really require proofs. We will use those rules however, to prove 3 derivative rules that are related to similar situations.

CONSTANT MULTIPLE RULE FOR DERIVATIVES

Given $g(x)=c\cdot f(x)$ where c is a constant, then $g'(x)=c\cdot f'(x)$

Proof: [Fill in the blanks to complete the proof]

$$g'(x)=\lim_{h\to 0}\frac{\square - g(x)}{h}$$

$$=\lim_{h\to 0}\frac{\square - c\cdot f(x)}{h}$$

$$=\lim_{h\to 0} c\cdot\left[\frac{\square - \square}{h}\right]$$

$$=c\cdot\lim_{h\to 0}\left[\frac{\square - \square}{h}\right]=c\cdot f'(x)$$

Q.E.D.

SUM RULE FOR DERIVATIVES

Given $j(x) = f(x) + g(x)$ where c is a constant,
then $j'(x) = f'(x) + g'(x)$

Proof: [Fill in the blanks to complete the proof]

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - \boxed{}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + \boxed{} - \left[f(x) - \boxed{} \right]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{\boxed{} - \boxed{}}{h}$$

$$= f'(x) + \boxed{}$$

Q.E.D.

DIFFERENCE RULE FOR DERIVATIVES

Given $j(x) = f(x) - g(x)$ where c is a constant,
then $j'(x) = f'(x) - g'(x)$

Proof: Use the proof of the sum rule as a guide to prove the difference rule.

PRACTICE

Given the following functions:

$$f(x) = 4x^2 + 6x - 8$$
$$g(x) = 5x^3 - 10x^2$$

1. (A) Find $f'(x)$ using the derivative rules discussed in section C.

(B) Find $g'(x)$ using the derivative rules discussed in section C.

2. (A) Find $f'(x)$ using the definition of derivative to verify your answer for 1(A).

(B) Find $g'(x)$ using the definition of derivative to verify your answer for 1(B).

<u>SECTION D</u>: DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

Let's take a look at the derivative of the following function $f(x) = e^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$ Substitution

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$ Product Rule for Exponents

$$= \lim_{h \to 0} e^x \cdot \left[\frac{e^h - 1}{h}\right]$$ Factoring

We can treat e^x like a constant here since h is the variable in this limit.

$$= e^x \cdot \lim_{h \to 0} \left[\frac{e^h - 1}{h}\right]$$ Constant Multiple Rule for Limits

Now to complete this derivative we need to address $\lim_{h \to 0} \left[\frac{e^h - 1}{h}\right]$ as this is not resolved easily using the algebraic means that we learned in part 3. So, let's take a look at this limit a little more closely. Fill in the table below.

h	$\frac{e^h - 1}{h}$
–1	
–0.5	
–0.1	
–0.01	
–0.001	
–0.00001	
0	UNDEFINED
0.00001	
0.001	
0.01	
0.1	
0.5	
1	

Based on your results from the previous page, what is the value of the following limit?

$$\lim_{h\to 0}\left[\frac{e^h-1}{h}\right]=$$

Now, with this result in mind, and the steps given on the previous page, complete the derivative rule for the natural exponential function below.

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

Given $f(x)=e^x$, then $f'(x)=$________

<u>SECTION E</u>: SUMMARY & ALTERNATE NOTATION

Up to this point, to indicate a derivative we have been using a notation like the following.

$$f(x)=x^2 \quad \Rightarrow \quad f'(x)=2x$$

Well, there is a quicker notation for expressing that a derivative is to be taken. The following statement is performing the same derivative that was done above.

$$\frac{d\left(x^2\right)}{dx}=\frac{d}{dx}\left(x^2\right)=2x$$

[The notation in the middle is usually the one used most often, the far right-hand side is, of course, the result of the derivative.]

In Calculus, *d* stands for, "a change in." This notation again, indicates that this is a rate of change, which is what we discussed in part 3 as the origin of a derivative.

On the next page, complete the rules that you have learned in this lesson.

CONSTANT RULE: $\frac{d}{dx}(c) =$, if c is a real-number constant.

POWER RULE: $\frac{d}{dx}\left(x^n\right) =$, if n is a real-number exponent.

CONSTANT MULTIPLE RULE:

$$\frac{d}{dx}\left[c \cdot f(x)\right] = \boxed{} \cdot \frac{d}{dx} f(x)$$, if c is a real-number constant.

SUM RULE: $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \boxed{\phantom{\frac{d}{dx} g(x)}}$

DIFFERENCE RULE: $\frac{d}{dx}\left[f(x) - g(x)\right] =$

NATURAL EXPONENTIAL RULE:

$$\frac{d}{dx}\left(e^x\right) =$$

EXERCISES

Find the derivatives of the following functions, using the derivative rules learned in this lesson.

1. $f(x) = 10x^7$

2. $f(x) = 7e^x$

3. $f(x) = \dfrac{12}{x^3}$

4. $f(x) = \sqrt{5}$

5. $f(x) = \sqrt[5]{x^3}$

6. $f(x) = \dfrac{8}{\sqrt{x}}$

7. $f(x) = 4x^3 - 7x^2 + 8x - 9$

8. $f(x) = 6x^2 + 4e^x - \dfrac{8}{x}$

9. $f(x) = \sqrt[3]{x^2} + \dfrac{1}{\sqrt[3]{x^2}}$

10. $f(x) = \dfrac{6x^2 + 4x + 3}{\sqrt{x}}$ ← You should carry through the division on this one first.

11. $f(t) = \dfrac{1}{6}t^6 - 3t^4 + t$

12. $f(t) = 5t^{-3/5}$

The Second Derivative

Given a function $f(x)$, the second derivative of $f(x)$ is given by the following formula:

$$f''(x) = \frac{d}{dx}\left[f'(x) \right]$$

Therefore, finding the second derivative simply means finding the derivative of a derivative.

Find the second derivative of the following functions.

13. $f(x) = e^x - 7x^2$

14. $f(x) = 15x^2$

15. $f(x) = 8x^3 + 4x - 11$

16. $f(x) = 2x - 5x^{3/4}$

17. $f(t) = \frac{9t^5 - 4t^2 + 3t}{t^3}$

18. $f(t) = \sqrt[3]{t^2} + 2\sqrt{t^3}$

Derivatives and Motion

We discussed in part 3 that the derivative of a distance function is a velocity function. In a similar fashion, an **acceleration** function is the result of the derivative of a velocity function. Therefore, the following is true:

> Given $d(t)$ is a function that gives the distance or position of an object at time *t*.
>
> $v(t) = d'(t)$ is the velocity function.
>
> $a(t) = v'(t) = d''(t)$ is the acceleration function.

Given the information above, complete the following problems.

19. The equation of the motion of a particle is $d(t) = 4t^3 - 12t^2$, where d is given in meters and t is in seconds. Find the following:

(a) The velocity and acceleration as functions of *t*.

(b) The acceleration after 2 seconds.

(c) The acceleration when the velocity is 0 and $t > 0$.

20. The equation of the motion of a particle is $d(t) = t^3 - 6t^2 + 4t + 1$, where d is given in feet and t is in minutes. Find the following:

(a) The velocity and acceleration as functions of *t*.

(b) The acceleration after 1 second.

(c) The velocity when the acceleration is 0.